Radiation from a Uniformly Accelerated Charge and the Equivalence Principle

Stephen Parrott
Department of Mathematics and Computer Science
University of Massachusetts at Boston
100 Morrissey Blvd.
Boston, MA 02125
USA

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Abstract

We argue that purely local experiments can distinguish a stationary charged particle in a static gravitational field from an accelerated particle in (gravity-free) Minkowski space. Some common arguments to the contrary are analyzed and found to rest on a misidentification of "energy".

1 Introduction

It is generally accepted that any accelerated charge in Minkowski space radiates energy. It is also accepted that a stationary charge in a static gravitational field (such as a Schwarzchild field) does *not* radiate energy. It would seem that these two facts imply that some forms of Einstein's Equivalence Principle do not apply to charged particles.

To put the matter in an easily visualized physical framework, imagine that the acceleration of a charged particle in Minkowski space is produced by a tiny rocket engine attached to the particle. Since the particle is radiating energy which can be detected and used, conservation of energy suggests that the radiated energy must be furnished by the rocket — we must burn more fuel to produce a given accelerating worldline than we would to produce the same worldline for a neutral particle of the same mass. Now consider a stationary charge in Schwarzchild space-time, and suppose a rocket holds it stationary relative to the coordinate frame (accelerating with respect to local inertial frames). In this case, since no radiation is produced, the rocket should use the same amount of fuel as would be required to hold stationary a similar neutral particle. This gives an experimental test by which we can determine *locally* whether we are accelerating in Minkowski space or stationary in a gravitational field — simply observe the rocket's fuel consumption. (Further discussion and replies to anticipated objections are given in Appendix 1.)

Some authors (cf. [3]) explain this by viewing a charged particle as inextricably associated with its electromagnetic field. They maintain that since the field extends throughout all spacetime, no measurements on the particle can be considered truly local. To the present author, such assertions seem to differ only in language from the more straightforward: "The Equivalence Principle does not apply to charged particles".

Other authors maintain that the Equivalence Principle *does* apply to charged particles. Perhaps the most influential paper advocating a similar view is one of Boulware [2], an early version of which formed the basis for the treatment of the problem in Peierls' book [10]. This paper claims to resolve "the equivalence principle paradox" by establishing that "all the radiation [measured by a freely falling observer] goes into the region of space time inaccessible to the co-accelerating observer."

A recent paper of Singal [8] claims that there is no radiation at all. Singal's argument, which we believe flawed, is analyzed in a forthcoming paper [7].

The present work analyzes the problem within Boulware's framework but reaches different conclusions. He shows that the Poynting vector vanishes in the rest frames of certain co-accelerating observers and concludes from this that

> "in the accelerated frame, there is no energy flux, ... , and no radiation".

Singal [8] rederives a special case of this result (his equation (7) on page 962), and concludes that "there are no radiation fields for a charge supported in a a gravitational field, in conformity with the strong principle of equivalence.

We obtain a similar result by other means in Appendix 3, but interpret it differently. We believe that the above quote of [2] incorrectly identifies the "radiated energy in the accelerated frame", and therefore does not resolve what he characterizes as a "paradox".

Also, we do not think there is any "paradox" remaining, unless one regards the inapplicability of the Equivalence Principle to charged particles as a "paradox". Even if the Equivalence Principle does not apply to charged particles, no known mathematical result or physical observation is contradicted.

2 What is "energy"?

The identification of "energy" in Minkowski or Schwarzchild spacetime may seem obvious, but there is a subtlety hidden in Boulware's formulation. This section examines this issue with the goal of clearly exposing the subtlety.

To deserve the name "energy", a quantity should be "conserved". The following is a well-known way to construct a conserved quantity from a zero-divergence

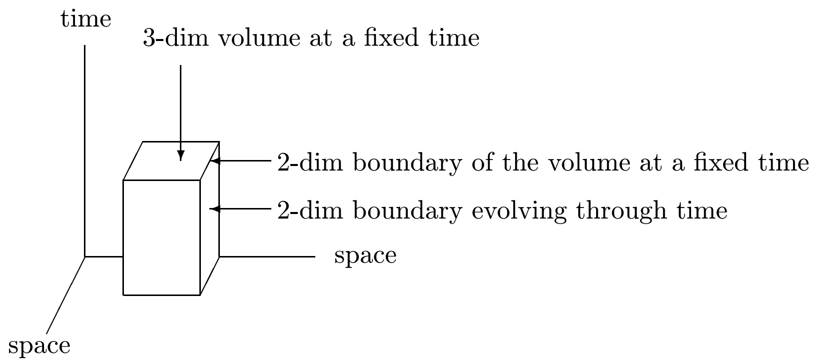

Figure 1: One space dimension is suppressed. The "top" and "bottom" of the box represent three-dimensional spacelike volumes; the "sides" represent two-dimensional surfaces moving through time; the interior is four-dimensional.

symmetric tensor $T = T^{ij}$ and a Killing vector field $K = K^i$ on spacetime. Form the vector $v^i := T^{i\alpha} K_\alpha$ (repeated indices are summed and usually emphasized by Greek and ":=" means "equals by definition"), and note that its covariant divergence ${v^\alpha}_{|\alpha}$ vanishes ([11], p. 96).

By Gauss's theorem, the integral of the normal component of v over the three-dimensional boundary of any four-dimensional region vanishes.[1] Such a region is pictured in Figure 1, in which one space dimension is suppressed. The particular region pictured is a rectangular "box" with spacelike "ends" lying in the constant-time hyperplanes $t = t_1$ and $t = t_2$ and time-like "sides". (We use t as a time coordinate and assume that it is, in fact, timelike.) The "end" corresponding to time t_i, $i = 1, 2$, represents a three-dimensional region of space at that time. The integral of the normal component of v over the end corresponding to $t = t_2$ is interpreted as the amount of a "substance" (such as energy) in this region of space at time t_2. The integral of the normal component over the sides is interpreted as the amount of the substance which leaves the region of space between times t_1 and t_2. Thus the vanishing of the integral over the boundary expresses a law of conservation of the substance. Similar interpretations hold even if the boundary of the region is "curved" and does not necessarily lie in constant coordinate surfaces.

We shall take as T^{ij} the energy-momentum tensor of the retarded electromagnetic field produced by a charged particle whose worldline is given. That is, if $F = F^{ij}$ is the electromagnetic field tensor, then

$$T^{ij} := F^{i\alpha} {F_\alpha}^j - (1/4) F^{\alpha\beta} F_{\alpha\beta} g^{ij} \quad , \tag{1}$$

where g_{ij} is the spacetime metric tensor. Given T, to every Killing vector field

[1] When there are points at which the boundary has a lightlike tangent vector, this must be interpreted sympathetically; see [5], Section 2.8 for the necessary definitions. However, we shall only need to integrate over timelike and spacelike surfaces, on which the concept of "normal component" is unambiguous.

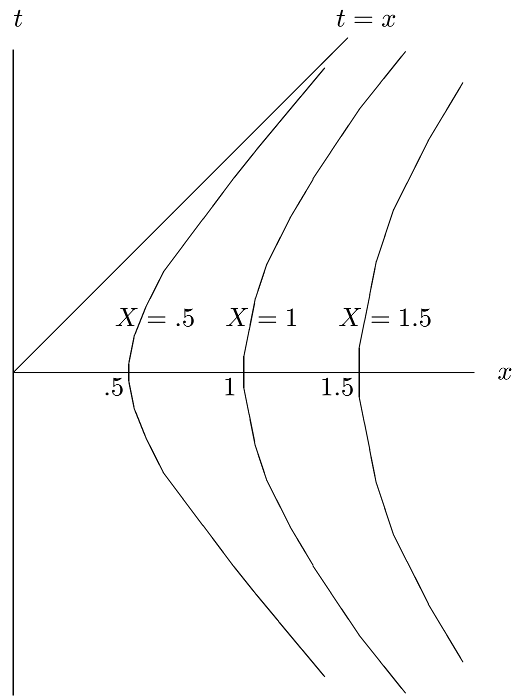

Figure 2: The orbits for the flow of the one-parameter family of boosts (3).

K corresponds a conserved scalar quantity as described above. We have to decide which such quantity deserves the name "energy".

In Minkowski space, the metric is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad , \tag{2}$$

and there seems no question that the energy is correctly identified as the conserved quantity corresponding to the Killing vector ∂_t generating time translations. (We use the differential-geometric convention of identifying tangent vectors with directional derivatives.) If this were not true, we would have to rethink the physical interpretation of most of the mathematics of contemporary relativistic physics. Translations in spacelike directions similarly give Killing vectors whose corresponding conserved quantities are interpreted as momenta in the given directions.

There are other Killing vector fields which are not as immediately obvious. For example, consider the Killing field corresponding to the flow of the one-parameter family $\lambda \mapsto \phi_\lambda(\cdot, \cdot, \cdot, \cdot)$ of Lorentz boosts

$$\phi_\lambda(t, x, y, z) := (t \cosh \lambda + x \sinh \lambda, t \sinh \lambda + x \cosh \lambda, y, z) \quad . \tag{3}$$

The orbits of this flow (curves obtained by fixing t, x, y, z and letting λ vary) are pictured in Figure 2. For fixed y, z, they are hyperbolas with timelike tangent vectors. Any such hyperbola is the worldline of a uniformly accelerated particle.

On any orbit, the quantity

$$X := (t\sinh\lambda + x\cosh\lambda)^2 - (t\cosh\lambda + x\sinh\lambda)^2 = x^2 - t^2$$

is constant, and its value is the orbit's x-coordinate at time $t = 0$. Thus an orbit is the worldline of a uniformly accelerated particle which had position $x = X$ at time $t = 0$.

Such an orbit can conveniently be described in terms of X as the locus of all points $(X\sinh\lambda, X\cosh\lambda, y, z)$, as λ varies over all real numbers. The tangent vector of such an orbit is

$$\partial_\lambda := (X\cosh\lambda, X\sinh\lambda, 0, 0) \quad .$$

This is the Killing vector field, expressed in terms of X and λ. Its length is X, so that a particle with this orbit has its proper time τ given by

$$\tau = \lambda X \quad , \tag{4}$$

its four-velocity ∂_τ is

$$\partial_\tau = \frac{1}{X}\partial_\lambda \quad , \tag{5}$$

and its proper acceleration is $1/X$.

The conserved quantity corresponding to the Killing vector ∂_λ has no recognized name, but it does have a simple physical interpretation which will be given below. We then argue that it is this quantity which [2] (p. 185) identifies (mistakenly, in our view) as the relevant "energy flux" in the accelerated frame.

3 Energy in static space-times

Consider a static spacetime whose metric tensor is

$$ds^2 = g_{00}(x^1, x^2, x^3)(dx^0)^2 + \sum_{I,J=1}^{3} g_{IJ}(x^1, x^2, x^3)dx^I x^J \quad . \tag{6}$$

The important feature is that the metric coefficients g_{ij} do not depend on the timelike coordinate x^0, so that ∂_{x^0} is a Killing field.

Another way to say this is that the spacetime is symmetric under time translation. In general, the flow of a Killing field can be regarded as a space-time symmetry. The symmetry of time translation was obvious from looking at the metric, but for some metrics there may exist less obvious, "hidden" symmetries. An example is the Minkowski metric (2), which possesses symmetries corresponding to one-parameter families of boosts which might not be obvious at first inspection.

Consider now the most important spacetime after Minkowski space, the Schwarzchild space-time with metric tensor

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \tag{7}$$

It can be shown ([11], Exercise 3.6.8) that the only Killing vector fields K are linear combinations of ∂_t and an "angular momentum" Killing field $A = K_\theta(r, \theta, \phi)\partial_\theta + K_\phi(r, \theta, \phi)\partial_\phi$, where K_θ and K_ϕ satisfy some additional conditions which are unimportant for our purposes. The fields ∂_t and A commute, as do their flows. In other words, the only Killing symmetries of Schwarzchild spacetime are the expected ones arising from rotational and time invariance: there are no hidden Killing symmetries.

In this situation, the only natural mathematical candidate for an "energy" is the conserved quantity corresponding to the Killing field ∂_t; for one thing, it is the only rotationally invariant choice. It is also physically reasonable in our context of analyzing the motion and fields of charged particles. If we surround a stationary charged particle[2] by a stationary sphere which generates a three-dimensional "tube" as it progresses through time, the integral of the normal component of T^{0i} over the tube between times t_1 and t_2 physically represents the outflow of the conserved quantity corresponding to ∂_t between these times. When the calculation is carried out, it is seen to be the same as integrating the normal component of the Poynting vector $\mathbf{E} \times \mathbf{B}/4\pi$ over the sphere and multiplying by a factor proportional to $t_2 - t_1$. It is usually assumed that the field produced by a stationary charged particle may be taken to be a pure electric field, and Appendix 3 proves this under certain auxiliary hypotheses. In other words, $\mathbf{B} = 0$, so the integral vanishes, and there is no "radiation" of our conserved quantity. We *expect* no energy radiation; otherwise we would be able to garner an unlimited amount of "free" energy, since it takes no energy to hold a particle stationary in a gravitational field.

Thus it seems eminently reasonable in this situation to identify the conserved quantity associated with ∂_t with the energy. We expect a conserved "energy", this is the only natural mathematical candidate, and its physical properties turn out to be reasonable.

However, these arguments lose force when hidden symmetries exist. Consider a metric

$$ds^2 = c(x)^2dt^2 - dx^2 - dy^2 - dz^2 \quad . \tag{8}$$

Here $c(x)$ represents the x-dependent speed of light as observed from the coordinate frame. Such a metric corresponds to a pseudo-gravitational field in the x-direction. By a "pseudo" gravitational field we mean that a stationary particle has a worldline which is accelerated in the x-direction, but the Riemann tensor may happen to vanish for some functions $c(\cdot)$, in which case there is no

[2]By a "stationary" particle we mean one whose worldline is $x^0 \mapsto (x^0, c^1, c^2, c^3)$ relative to the static coordinate frame with respect to which the metric is (7), where the c^i are constants independent of x^0.

curvature of space-time and no true gravitational field. It is well known that when the Riemann tensor vanishes, spacetime may be metrically identified with a piece of Minkowski space.

Routine calculation shows that the only nonvanishing connection coefficients are, in an obvious notation,

$$\Gamma^t_{tx} = \Gamma^t_{xt} = \frac{c'}{c} \quad , \qquad \Gamma^x_{tt} = c'c \quad .$$

The four-velocity u of a stationary particle is $u = c^{-1}\partial_t$, so a stationary particle has acceleration $(D_u u)^k = u^\alpha \partial_\alpha u^k + \Gamma^k_{\alpha\beta} u^\alpha u^\beta$ given by

$$D_u u = \frac{c'}{c}\partial_x \quad , \tag{9}$$

That is, the acceleration is in the x-direction with a magnitude given by the relative rate of change of c in the x-direction. This acceleration $D_u u$ is what we mean by "acceleration with respect to local inertial frames".

It might seem reasonable, even natural, to identify the conserved quantity associated with ∂_t with energy, in analogy with Schwarzchild spacetime. However, the reasonableness of such an identification must ultimately be justified by its mathematical and physical consequences. We shall argue that such an identification is sometimes inappropriate.

The "obvious" Killing symmetries of (8) are those associated with time translation, translations in spatial directions perpendicular to the x-axis, and rotations about the x-axis. Only for very special choices of $c(\cdot)$ will there exist other, "hidden" symmetries. One such choice yields the following metric, in which for later purposes we replace the coordinate symbol x by X and t by λ:

$$ds^2 = X^2 d\lambda^2 - dX^2 - dy^2 - dz^2 \quad . \tag{10}$$

The Riemann tensor vanishes for this spacetime, and it can be identified with a piece of Minkowski space. If t, x, y, z denote the usual Minkowski coordinates with metric given by (2), then this identification is:

$$\begin{aligned} t &= X \sinh \lambda \\ x &= X \cosh \lambda \quad . \end{aligned}$$

The part of Minkowski space covered by the map $\lambda, X, y, z \mapsto t, x, y, z$ consists of the region $|x| > |t|$, but we will only be concerned with the smaller region $x > |t|$.

The coordinates λ, X, y, z for this portion of Minkowski space are known as *Rindler* coordinates ([12], Section 8.6). They are also sometimes known as *elevator* coordinates because we shall see below that X, y, z may be regarded as space coordinates as seen by occupants of a rigidly accelerated elevator. Boulware [2] uses τ in place of λ for the timelike coordinate. We prefer λ because it seems more natural to reserve $\tau = \lambda X$ for the proper time on the worldlines of points of the elevator.

For constant y and z, a curve $X = constant$ is the orbit of $t = 0, x = X$ under the flow (3). This curve is also the worldline of a uniformly accelerated particle with proper acceleration $1/X$. The set of all such curves for all X, y, z may be regarded as the worldlines of a collection of uniformly accelerated observers all of whom are at rest in the Minkowski frame at time $t = 0$.

The Rindler coordinates X, y, z specify the particular worldline in the collection. The spatial distance between two points with the same Rindler "time" coordinates say λ, X_1, y_1, z_1 and λ, X_2, y_2, z_2, is just the ordinary Euclidean distance $[(X_2 - X_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$. Moreover, the corresponding spatial displacement vector is orthogonal to the worldlines of constant X, y, z. This says that an observer following such a worldline sees at any given moment other such worldlines at a constant distance in his rest frame at that moment. Thus we may take a collection of such worldlines and imagine connecting them with rigid rods (the rods can be rigid because the proper distances are constant), obtaining a rigid accelerating structure which we might call an "elevator".

However, it would be misleading to call it a *uniformly* accelerating elevator. Though every point on it is uniformly accelerating, the magnitude $1/X$ of the uniform acceleration is different for different points. Because of this, the everyday notion of a uniformly accelerating elevator gives a potentially misleading physical picture. A more nearly accurate picture is obtained by thinking of each point of the elevator as separately driven on its orbit through Minkowski space by a tiny rocket engine. Observers moving with the elevator experience a pseudo-gravitational force which increases without limit as the "floor" of the elevator at $x = 0$ is approached; observers nearer the floor need more powerful rockets than those farther up.

We have two ways to view the physics of such an elevator. On the one hand, since the elevator *is* a subset of Minkowski space, we can transform the well-understood physics of Minkowski space into elevator coordinates to derive what residents of the elevator should observe. In particular, if a particle of charge q is situated at $X = 1$, say, its motion being driven by a tiny rocket attached to it, then the energy required by the rocket per unit proper time would be the energy required for an uncharged particle of the same mass plus the radiated energy, the proper-time rate of radiated energy being $(2/3)q^2$ as required by the Larmor Law for proper acceleration $1/X = 1$.

A second approach would be to emphasize the analogy of the metric (10) with the Schwarzchild metric (7), interpreting the conserved quantity corresponding to ∂_λ as the "energy". We want to emphasize that *these two approaches are essentially different and yield different physical predictions.*

We'll see below that the second approach (which seems similar to that of [2]) yields a conserved quantity whose integral over the "walls" of (say) a spherical elevator surrounding the particle is *zero*. That is, there is *no* radiation of this conserved quantity, which we'll call the "pseudo-energy" to distinguish it from the above Minkowski energy. If we interpreted this pseudo-energy as energy radiation as seen by observers in the elevator (such as the pilot of the rocket accelerating the charge), then by conservation of energy we should conclude that no additional energy is required by the rocket beyond that which would be required to accelerate an uncharged particle of the same mass.

This is a different physical prediction than the corresponding prediction based on Minkowski physics, and difference between the two predictions is in principle experimentally testable. It is precisely at this point that we differ from [2]. That reference does distinguish between the Minkowski energy and the pseudo-energy, but it gives the impression that they are somehow the same "energy" measured in different coordinate systems. We think it is worth emphasizing that they are not the same energy measured in different systems; instead, they are different "energies" derived from different Killing fields. The observation that the pseudo-energy radiation is zero does not validate the equivalence principle.

4 Discussion of calculation of radiation

We want to briefly discuss what we think is the physically correct way to calculate the energy radiated by an accelerated charge in Minkowski space. Almost everything we shall say is well known, but we want to present it in a way which will make manifest its applicability to the present problem. The analysis to be given does not apply to spacetimes other than Minkowski space for reasons which will be mentioned later. It does, of course, apply to a Rindler "elevator", since this is a subset of Minkowski space.

Suppose we are given the worldline of a (not necessarily uniformly) accelerated particle and a proper time τ. Surround the particle by a two-dimensional surface S_τ. It may be useful to think of S_τ as a sphere, but we don't assume any metrical properties for S, such as rigidity. All we assume is that S_τ surrounds the particle.

As the particle progresses on its worldline, let S_τ move with it in such a way that the particle is always surrounded. As proper time progresses from an initial value τ_1 to a later value τ_2, the surface S_τ generates a three-dimensional manifold $S(\tau_1, \tau_2)$ in Minkowski space which is customarily called a "tube", because it looks like a tube surrounding the worldline in a picture of Minkowski space in which one space dimension is suppressed.

The integral of the energy-momentum tensor $T = T^{ij}$ over this three-dimensional manifold will be denoted

$$\int_{S(\tau_1,\tau_2)} T^{i\alpha}\, dS_\alpha \quad . \tag{11}$$

The precise mathematical definition of (11) is discussed in detail in [5]. Since the definition entails summing vectors in different tangent spaces, it does not

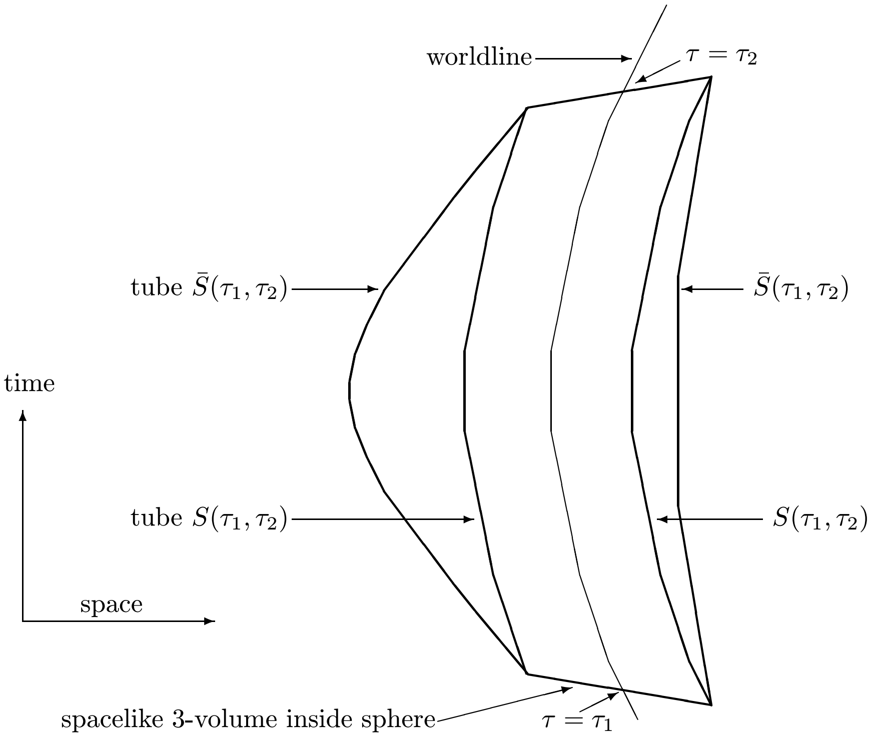

Figure 3: Two three-dimensional tubes which coincide at their ends.

make sense in general spacetimes, in which there is no natural identification of tangent spaces at different points.

The intuitive meaning is that for fixed i, we integrate the normal component of the vector $T^{i\alpha}$ over the tube, the integration being with respect to the natural volume element on the tube induced from Minkowski space. Physically, (11) is interpreted as the energy-momentum radiated through S_τ for $\tau_1 \leq \tau \leq \tau_2$. The energy radiated is (11) with $i = 0$.

Suppose we have two tubes, say S_τ and $\bar{S}_\tau$, which coincide at the initial and final proper times τ_1 and τ_2: $S_{\tau_1} = \bar{S}_{\tau_1}$ and $S_{\tau_2} = \bar{S}_{\tau_2}$. Such a situation is pictured in Figure 3, in which two space dimensions are suppressed. Taken together, they form the boundary of a four-dimensional region, and since T has vanishing divergence off the worldline,

$$\int_{S(\tau_1,\tau_2)} T^{i\alpha}\, dS_\alpha = \int_{\bar{S}(\tau_1,\tau_2)} T^{i\alpha}\, dS_\alpha \quad . \tag{12}$$

In other words, the calculated radiation is *independent of the tube*, so long as the tubes coincide at their ends. Put another way, no matter how the sphere distorts on its journey, (11) always produces the same numerical results for the radiated energy-momentum.

This leads to considerable conceptual and mathematical simplification in the important special case in which the particle is unaccelerated in the distant past, put into accelerated motion for a while, and re-enters an unaccelerated state in the distant future. We can take the ends of the tube as any convenient geometrical shape in the rest frame of the unaccelerated particle in the distant past (or future), say a sphere. The field energy-momentum inside the sphere is the infinite energy of the Coulomb field, which is discarded in a mass renormalization. It is unfortunate that the energy is infinite, but at least it is well understood and can be unambiguously calculated in this special case; this is the reason for insisting that the particle be unaccelerated in distant past and future.

Suppose the two-dimensional surrounding surface consists of the walls of an elevator with the charged particle at its center. Suppose the elevator is initially at rest and then both elevator and particle are gently nudged into rigidly accelerated motion in which the elevator is described in Rindler coordinates and the particle is at rest in the Rindler frame. The state of rigidly accelerated motion is then maintained for an arbitrarily long period, after which the acceleration is gently removed and the elevator enters a state of uniform motion thereafter. The result of the integral (11) for a spherical elevator of initial radius ϵ is well-known. Denoting the particle's four-velocity at proper time τ as $u(\tau)$, the proper acceleration as $a(\tau) := du/d\tau$, and $a^2 := a^\alpha a_\alpha \leq 0$, it is ([5], p. 160):

$$\int_{S(\tau_1,\tau_2)} T^{i\alpha}\, dS_\alpha \;=\; -\frac{2}{3}q^2 \int_{\tau_1}^{\tau_2} a^2(\tau) u^i(\tau)\, d\tau + \frac{q^2}{2\epsilon}[u^i(\tau_2) - u^i(\tau_1)] \quad , \tag{13}$$

where q is the particle's charge. The last term on the right is traditionally discarded in a mass renormalization. The energy component of the first term is *always positive.* We conclude that there is energy radiation through the walls of the elevator.

This energy radiation can be detected in several ways in an arbitrarily small elevator. First of all, if we believe in conservation of the usual Minkowski energy, the pilot of the rocket driving the charge will observe an additional fuel consumption when his payload is a charged particle *vis a vis* the corresponding identical motion of an uncharged particle, the additional fuel consumption being exactly the amount necessary to "pay" for the radiated energy. (However, the details of how this "borrowed" energy must be repaid may be controversial, as discussed in Appendix 1.)

A more fundamental way to meaure it, at least in principle, is to divide the elevator walls into a large number of small coordinate patches with an observer stationed on each patch. Instruct the observers to measure the fields, calculate the corresponding energy-momentum tensor, and approximate to arbitrary accuracy the energy component of the integral (11).

We want to emphasize that this is *not the same* as having each observer calculate his local energy outflow $\mathbf{n} \cdot (\mathbf{E} \times \mathbf{B}/4\pi)\Delta S \Delta\tau$ (where $\mathbf{n}$ is the outward unit normal vector to the wall in the observer's rest frame, ΔS the area of his patch, $\Delta\tau$ the increment in his proper time, and $\mathbf{E}$ and $\mathbf{B}$ his electric and magnetic fields, respectively), and finally adding up the total energy outflow of all the observers. For arbitrary motion (i.e. elevator allowed to distort), this last procedure would have no invariant meaning because each observer has his own private rest frame at each instant of his proper time. The "energy" obtained as the final result of this procedure would in general depend on the construction of the elevator. For instance, if on the same trip we had a small elevator surrounded by a larger one, there is no reason to suppose that the observers on the larger elevator would obtain the same number for "energy" radiation as those on the smaller. Neither number would be expected to be correlated in any way with the additional amount of energy to be furnished by the rocket for a charged versus uncharged payload.

In the procedure just described, the observers are not measuring "energy"; they are measuring something else. It may seem tempting to call it something like "energy as measured in the (curvilinear) elevator frame", but it is conceptually and experimentally distinct from the usual Minkowski energy. For arbitrary motion, it is not a conserved quantity and therefore probably does not deserve the name "energy". For the special case of an elevator with constant spatial Rindler coordinates, it does happen to be independent of the elevator's shape (in fact, it's zero for all!), but it is still not "energy" as the term is normally used. We'll show below that it is the conserved quantity corresponding to the Killing vector for ∂_λ; i.e. the quantity which we previously named the "pseudo-energy".

The pseudo-energy as physically measured by the procedure just described for a spherical elevator S of radius R in Rindler coordinates is mathematically given by the following integral in spherical Rindler coordinates R, θ, ϕ (which bear the same relation to rectangular Rindler coordinates X, y, z that ordinary spherical coordinates r, θ, ϕ bear to Euclidean coordinates x, y, z). In the integral, $u = u(\tau, R, \theta, \phi)$ denotes the four-velocity of the point of the elevator located at Rindler spherical coordinates R, θ, ϕ at its proper time τ (i.e. Rindler time coordinate $\lambda = \tau/X$), and $n = n(R, \theta, \phi)$ is the spatial unit normal vector to the sphere at the indicated point (i.e., n is orthogonal to u and normal to the sphere, so that in Rindler coordinates, $n = (0, \mathbf{n})$):

$$\text{pseudo-energy radiation} = \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, R^2 \sin\theta\, u_\alpha T^{\alpha\beta}(-n_\beta) \quad . \tag{14}$$

(The minus sign is because the spatial inner product is negative definite.)

Recall from (5) that $u = \partial_\tau = \partial_\lambda / X$, so that in Rindler coordinates in which $\partial_\lambda =: \partial_0$ is associated with the zero'th tensor index, $u_\alpha T^{\alpha\beta} n_\beta = X T^{0\beta} n_\beta = X \sum_{J=1}^{3} T^{0J} \mathbf{n}_J$. Recalling also from (4) that $\tau = X\lambda$ and that $K_0 = X^2$, we may rewrite (14) in Rindler coordinates as:

$$\text{pseudo-energy radiation} = \int_{\lambda_1}^{\lambda_2} d\lambda \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, R^2 \sin\theta \sum_{J=1}^{3} -K_\alpha T^{\alpha J} \mathbf{n}_J \quad , \tag{15}$$

with $\lambda_i := \tau_i / X$, $i = 1, 2$. Equation (15) demonstrates that (14) is actually computing the radiation of the conserved quantity corresponding to the Killing vector $K = \partial_\lambda$.

A sufficient condition for (15) to vanish is for $T^{0J} = 0$ for all spatial indices J in Rindler coordinates. Equation (IV.3), p. 185 of [2] establishes that $T^{0J} = 0$ and from this draws the conclusion that:

> "in the accelerated frame there is no energy flux, ... , and no radiation".

That $T^{0J} = 0$ is essentially the well-known "fact"[3] that a stationary charged particle in a static spacetime does not radiate energy, where "energy" is defined as the conserved quantity corresponding to translation by the formal time coordinate (in this case, λ) in this spacetime.

We agree with [2] that there is no radiation of the conserved quantity corresponding to the Killing vector ∂_λ, but we believe that this fact is irrelevant to questions concerning physically observed radiation and to questions about the applicability of the Equivalence Principle. Whether it is Minkowski energy radiation or pseudo-energy radiation which corresponds to energy that must be furnished by the driving forces is an experimental question. In principle, it could be settled by uniformly accelerating a large charge in a rocket and observing if more fuel were required than for a neutral payload of the same mass. We would bet that more fuel would be required, which would mean that Minkowski energy is the physically relevant "energy".

On the other hand, in the Schwarzchild spacetime (7), the energy corresponding to the analog ∂_t of ∂_λ is universally accepted as the physically relevant "energy". The spacetime (10) provides an interface between a Schwarzchild-type spacetime and Minkowski space within which questions about the Equivalence Principle can be conveniently addressed. If our hypothesis that Minkowski energy is the physically relevant "energy" in (10) is correct, then the vanishing of (15) not only does not validate the Equivalence Principle, but strongly suggests that it does *not* apply to charged particles. If we treat questions of radiation in the spacetime (10) in exactly the same way that such questions are treated in Schwarzchild space (7), then we are led to the probably incorrect conclusion that the rocket accelerating the charge does not require any extra fuel, since there is no radiation.

The assertion that $T^{0J} = 0$ for spatial indices J implies that "in the accelerated frame there is ... no [energy] radiation" merits further discussion because similar arguments are used by other authors, and we believe that language such as "energy radiation in the accelerated frame" encourages a subtle error. The 3-vector T^{0J} is the Poynting vector: $(T^{01}, T^{02}, T^{03}) = (\mathbf{E} \times \mathbf{B})/4\pi$, so that $T^{0J} = 0$ says that every elevator observer sees a zero Poynting vector. If we identify seeing a zero Poynting vector with seeing no energy radiation, then this says that no elevator observer sees any energy radiation, which seems to lead to the conclusion that there is no energy radiation "in the elevator frame".

[3] We put "fact" in quotes because although this assertion is often made, we know of no proof in the literature, and in fact, it seems unlikely that it has been proved. Appendix 3 discusses this problem and furnishes a proof under certain auxiliary hypotheses.

Of course, one could obtain this conclusion by taking the vanishing of the Poynting vector in the elevator frame to be the *definition* of "no energy radiation in the elevator frame", but we argue that such a definition would be physically inappropriate. In fact, this is the main point of this section: although each observer in a rigidly accelerating elevator surrounding the particle measures a vanishing Poynting vector *in his own private rest frame*, nevertheless, taken as a whole there is radiation through the elevator walls. Adding the (zero) energy fluxes measured by each observer on the wall to (incorrectly) conclude zero total energy radiation is an illegitimate operation because these energy fluxes refer to different rest frames.

5 Remarks on detecting energy radiation near a particle

Reference [2] (unlike [8]) does recognize that Minkowski energy radiation is nonzero but concludes that it cannot be detected within the elevator (and thence that there is no violation of the Equivalence Principle). It discusses and discards several possible methods to observe Minkowski radiation within the elevator. For example, "if one identifies the radiation by the $1/r$ dependence of the field along the light cone, one cannot ... remain within [the region covered by the elevator coordinates] and let r become large enough for the radiation field to dominate." This overlooks the fact that the field components are analytic functions off the worldline, and an analytic function is uniquely determined by its values on any open set, however small. To pick off the radiation terms that go to zero like $1/r$ as $r \to \infty$, we need only evaluate the field at a few points, which can be as close to the worldline as we want, and perform a few algebraic calculations to find the coefficients of the $1/r$ terms. For example, if we write the fields in terms of the retarded distance r_{ret}, the field components in a given direction from the retarded (emission) point are simple quadratic polynomials in $1/r_{ret}$, whose coefficients can be easily determined.

6 Conclusions

Does Einstein's Equivalence Principle hold for charged particles? We cannot definitively answer this because a mathematically precise statement of the "equivalence principle" seems elusive — most statements in the literature are not sufficiently definite to be susceptible of proof or disproof. However, we do conclude that most usual formulations seem not to hold in any direct and obvious way for charged particles.

We believe that [2], which is widely cited in contexts suggesting that its analysis supports the validity of the Equivalence Principle for charged particles, does not in fact validate any form of the Equivalence Principle. We argue that its conclusion that "in the accelerated frame, there is no energy flux, ... and no radiation", is correct only if "energy" is misidentified (in our view) as the conserved quantity associated with a one-parameter family of Lorentz boosts in Minkowski space, instead of with the one-parameter family of time translations.

Appendix 1: The relation of the Lorentz-Dirac equation to this problem

We anticipate that some readers may be uneasy about our assertion that a uniformly accelerated charge in gravity-free (i.e., Minkowski) space may be *locally* distinguished from a stationary charge in (say) Schwarzchild space-time by observing how much energy an external force, such as our fanciful rocket, must supply to maintain the worldline. Some may observe that in Minkowski space, the radiation reaction term in the Lorentz-Dirac equation vanishes, so one might think that no more energy would be required in either case than would be needed for a neutral particle of the same mass. This appendix discusses this point, which is important but peripheral to the main text.

The Lorentz-Dirac equation[4] for a particle of mass m and charge q in an external field $F = F^i{}_j$ is:

$$m\frac{du}{d\tau} = qF(u) + \frac{2}{3}q^2\left[\frac{da}{d\tau} + a^2u\right] \quad , \tag{16}$$

where τ is proper time, $u = u^i$ the particle's four-velocity, $a := du/d\tau$ its proper acceleration, $a^2 := a^\alpha a_\alpha$ and $F(u)^i := F^i{}_\alpha u^\alpha$.

The left side is the rate of change of mechanical energy-momentum, the term $qF(u)$ is the Lorentz force, and the remaining term $(2/3)q^2[da/d\tau + a^2u]$ is the "radiation reaction" term which describes the effect of the particle's radiation on its motion.

For a uniformly accelerated particle (i.e., a^2 is constant) moving in one space dimension, the radiation reaction term $(2/3)q^2[da/d\tau + a^2u]$ *vanishes identically.*[4] It is tempting to interpet this as implying that there is no physical radiation reaction for a uniformly accelerated charged particle, by which we mean that a rocket-driven uniformly accelerated charge requires no more energy from the rocket than an otherwise identical neutral charge. However, we believe such an interpretation is unlikely to be correct.

An obvious flaw in the argument just given is that it is inconsistent with usual ideas of conservation of energy. If we grant that the uniformly accelerated charge

[4]To see this, write $u = (\gamma, v\gamma, 0, 0)$ with v the velocity and $\gamma := (1 - v^2)^{-1/2}$, and let $w := (v\gamma, \gamma, 0, 0)$ be an orthogonal unit vector associated with the same spatial direction. By general principles, the proper acceleration a is orthogonal to u, so that $a = Aw$ for some scalar function A, and $A^2 = -a^2$ is constant. Since w is a unit vector, $dw/d\tau$ is orthogonal to w, and hence $da/d\tau = Adw/d\tau$ is a multiple of u. That the multiple is $-a^2$ can be determined by taking the inner product $u^\alpha da_\alpha/d\tau = d(u^\alpha a_\alpha)/d\tau - a_\alpha du^\alpha/d\tau = -a^2$.

does radiate energy into Minkowski space which can be collected and used, as nearly all modern authors ([8] excepted) seem to agree, then this radiated energy must be furnished by some decrease in energy of other parts of the system. Fulton and Rohrlich [19] suggest that it may somehow come from the field energy but give no proof. (Since the field energy in a spacelike hyperplane is infinite for a point electron, it's not clear what would constitute a proof.)

We look at the matter differently. All derivations of the Lorentz-Dirac equation are motivated by conservation of energy-momentum: the change of energy-momentum of the particle over a given proper-time interval should equal the energy-momentum furnished by the external forces driving the particle minus the radiated energy-momentum, assuming that it is legitimate to absorb infinite terms of a certain structure into a mass renormalization. Although this principle motivates the derivation, the final equation unfortunately does *not* guarantee such conservation of energy-momentum in general, but only in certain special cases. One such special case is when the particle is asymptotically free, meaning that its proper acceleration vanishes asymptotically in the infinite past and future.

Thus it's not clear that the equation should apply to a particle which is not asymptotically free, such as a particle which is uniformly accelerated for *all* time.[5] Since the equation doesn't guarantee conservation of energy-momentum for uniform acceleration for all time, the fact that the radiation reaction term vanishes implies nothing about the additional force which the rocket must furnish for perpetually uniformly accelerated motion.

But we should at least try to understand the case of a particle which is unaccelerated in the distant past, nudged into uniform acceleration, uniformly accelerated for a long time, and finally nudged back into an unaccelerated state. For this case, the Lorentz-Dirac equation *does* imply conservation of energy-momentum. However, since the radiation reaction term vanishes for the period of uniform acceleration, the equation implies that *all the radiation energy must be furnished at the beginning and ending of the trip*, while the particle is nudged into or out of its uniformly accelerated state.

In other words, if we believe in the Lorentz-Dirac equation, we need to add a bit of energy to start the uniform acceleration, and thereafter the radiation, which can persist for an arbitrarily long time and add up to an arbitrarily large amount, is "free" until the end of the trip. In effect, we can "borrow" an arbitrarily large amount of radiated energy (which in principle can meanwhile be collected and used by other observers in Minkowski space), so long as we pay it back at the end of the trip. Although there is no logical contradiction here, this is hard to accept physically, and seems one of many good reasons to question the Lorentz-Dirac equation.

Most of the above issues are only peripherally relevant to the present work,

[5] Actually, the equation is controversial even for asymptotically free particles, but that brings up issues outside the scope of this article. The reader can find more information in [14], [5], [6], [15].

and we present them only to dispel potential confusion. The point is that the vanishing of the radiation reaction term does *not* imply that the rocket accelerating the charged particle in Minkowski space does not have to furnish the radiation energy. The rocket almost certainly does have to supply this energy, and this gives a local experiment which distinguishes certain accelerated motion in Minkowski space from similar motion in Schwarzchild space.

Appendix 2: The equation of motion of a charged rocket

A noted expert in the field raised the following interesting objection to the discussion of Appendix 1 in an earlier version of this paper. Consider a charged rocket which undergoes a modest uniform acceleration g (one gravity, say) from just after an inital time τ_i to just before a final time τ_f. More precisely, the rocket is at rest in some Lorentz frame (the *initial frame*) up to some initial proper time τ_i, nudged into uniform acceleration over a small proper time interval $[\tau_i, \tau_i + \delta]$, uniformly accelerated up to proper time $\tau = \tau_f - \delta$, nudged back into an unaccelerated state over the interval $[\tau_f - \delta, \tau_f]$, to remain unaccelerated for $\tau > \tau_f$.

He presented a simple estimate showing that the energy required to accomplish the final deceleration, *as measured in the final rest frame* at $\tau = \tau_f$, is modest and *independent of the period of uniform acceleration.* This can be anticipated without calculation, since from the viewpoint of the final rest frame, going backwards in time from $\tau = \tau_f$ to $\tau = \tau_f - \delta$ only requires nudging the rocket back up to a modest uniform acceleration, and this cannot not require an unbounded energy change. Thus it would seem that from the point of view of the rocket's pilot, only a modest amount of fuel must be burned to start the acceleration at the beginning of the trip and stop it at the end, with no excess fuel (relative to an uncharged rocket) required during the period of uniform acceleration (which can be arbitrarily long).[6]

The modest amount of energy used in the final frame (along with its associated momentum) can Lorentz-transform into a large amount of energy in the initial rest frame at $\tau = \tau_i$, so there is no apparent violation of conservation of energy from the standpoint of the initial frame. However, we can obtain what might appear to be a violation if we imagine reversing the proper acceleration a at the final time τ_f in a time-symmetric way (i.e., $a(\tau_f + \sigma) = -a(\tau_f - \sigma)$)

[6] This was produced in evidence for the widely held belief (which we think incorrect) that there is no radiation reaction for a uniformly accelerated charge in Minkowski space. This line of reasoning suggests that we could allow the uniform acceleration to continue indefinitely without using any more fuel. (By extension, perpetual uniform acceleration would presumably require no fuel at all.) That would violate conservation of energy, assuming that the radiation energy is physically accessible, but proponents of this view sometimes use arguments similar to those criticized in Section 5 to assert that radiation cannot be observed within a Rindler elevator.

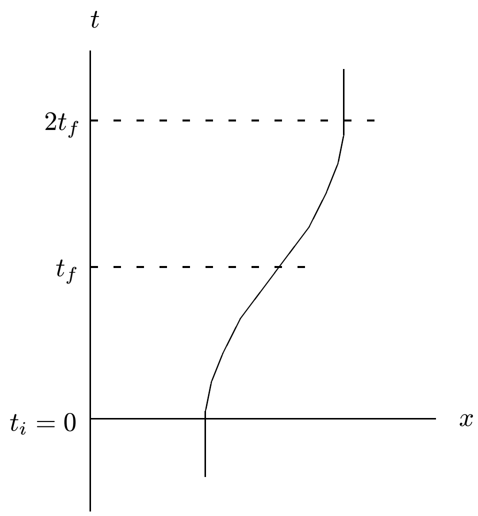

Figure 4: The worldline of a particle at rest up to time t_i and uniformly accelerated from time $t_i + \delta$ to $t_f - \delta$, where δ is the length of a small time interval during which the particle is nudged into or out of uniform acceleration. At time t_f the acceleration is reversed in a time-symmetric way so as to bring the particle back to rest at time $2t_f$.

to eventually bring the rocket back to rest in the initial frame at $\tau = 2\tau_f$, as depicted in Figure 4. The expert's estimate shows that the excess fuel used over the entire trip from $\tau = \tau_i$ to $\tau = 2\tau_f$ is modest and independent of the duration of the uniform acceleration. At the beginning and end of the trip the rocket is at rest in the initial frame, so the energy of the radiation plus the exhaust should equal the rest-mass loss of the rocket (fuel used). If the loss of rest mass is finite and independent of the duration of the uniform acceleration (hence independent of the amount of energy radiation), we have a violation of conservation of energy in the initial frame.

The situation was clarified by actually solving the equation of motion for the radiating rocket, using the Lorentz-Dirac radiation reaction expression. It turns out that with a fixed amount of initial fuel, one cannot obtain an arbitrarily large period of uniformly accelerated motion (i.e., arbitrarily large τ_f) unless one allows the rocket mass to go negative. Put another way, a charged rocket will run out of fuel if it uniformly accelerates long enough, so our time-symmetric motion is impossible with fixed initial fuel and arbitrarily large τ_f. This is in contradistinction to a uniformly accelerated uncharged rocket which can accelerate forever, assuming that all of its mass can be used as fuel.

In retrospect, this conclusion seems natural and the analysis leading to it elementary, but I found the expert's objection sufficiently troubling to feel it necessary to actually work it out. Having done so, perhaps including it here may save readers with similar questions some work.

The rocket will always move in the positive x-direction, and the other two constant space coordinates will be suppressed. If its initial-frame velocity is v, its initial-frame "rapidity" θ is defined by $\theta := \tanh^{-1}(v)$. Then its four-velocity u is given in initial-frame coordinates by

$$u = (\cosh\theta, \sinh\theta) \quad .$$

The *scalar proper acceleration* A is defined by $du/d\tau = Aw$ where w is the unit vector

$$w := (\sinh\theta, \cosh\theta)$$

orthogonal to u. The (four-vector) proper acceleration is $a := Aw$.

The scalar proper acceleration is related to the rapidity by $A = d\theta/d\tau$. In particular for constant scalar proper acceleration $A(\tau) \equiv g$, we have $\theta(\tau) = g\tau + \theta(0)$.

The Lorentz-Dirac expression for the proper-time rate of energy-momentum radiation of a charge q is:

$$\begin{aligned} \text{rate of energy-momentum radiation} &= (2q^2/3)(da/d\tau + a^2 u) \\ &= -(2q^2/3)(dA/d\tau)w \quad . \end{aligned} \tag{17}$$

The second line follows from the first in a fashion similar to that of the first footnote in Appendix 1. We eliminate the constant factor by choosing units so that $2q^2/3 = 1$.

Let $m(\tau)$ denote the rocket's rest mass, so that $-dm/d\tau$ is the rate of ejection of rest mass into the exhaust. (This is not the same as the rate at which the exhaust acquires rest mass, as will be apparent from the expressions to follow. Rest mass is not conserved in general.)

There are two parameters which can be used to control the rocket's worldline: the exhaust velocity and the rate $-dm/d\tau$ of ejection of rest mass into the exhaust. The analysis to follow assumes that the exhaust velocity as seen from the rocket is always constant and that $dm/d\tau$ is varied so as to produce the desired worldline. The rocket is always moving to the right in the initial frame, so the exhaust is always moving left. We allow $dm/d\tau$ to have either sign. A positive $dm/d\tau$ means that the rocket is taking on mass. This could physically be accomplished by shooting bullets into it from the right, with the bullets constituting the "exhaust".

Let $-\nu$ denote the exhaust rapidity in the rocket's instantaneous rest frame. That is, ν is positive, and an exhaust particle has velocity $\tanh(-\nu)$ in the this rest frame. Then the exhaust's four-velocity is:

$$\text{four-velocity of exhaust} = u\cosh\nu - w\sinh\nu \quad . \tag{18}$$

Let $\rho = \rho(\tau)$ denote the proper-time rate at which the exhaust rest mass is increasing. Let $Rw = R(\tau)w(\theta(\tau))$ denote the proper-time rate at which electromagnetic energy-momentum is being emitted. It is assumed that this rate is a multiple of w because the above Lorentz-Dirac expression is of this form. It is convenient to allow arbitrary R because this enables us to simultaneously treat the case of an uncharged rocket by setting $R \equiv 0$. The uncharged case is also worked out in [17], in a similar fashion with identical results.

The equation of energy-momentum balance is:

$$\begin{aligned} 0 &= \frac{d(mu)}{d\tau} + (u\cosh\nu - w\sinh\nu)\rho + Rw \\ &= (\frac{dm}{d\tau} + \rho\cosh\nu)u + (mA - \rho\sinh\nu + R)w \quad . \end{aligned} \tag{19}$$

The first term in the first line is the proper-time rate of change of energy-momentum of the rocket, the second term the proper-time rate at which the exhaust is acquiring energy-momentum, and the third the proper-time rate of energy-momentum radiation.

Since u and w are orthogonal, the last line separates into two independent equations:

$$\rho = -\frac{1}{\cosh\nu}\frac{dm}{d\tau} \quad , \tag{20}$$

and

$$mA - \rho\sinh\nu + R = 0 \quad . \tag{21}$$

Equation (20) may be regarded as defining ρ, and then (21) becomes, setting $\lambda := 1/\tanh\nu$:

$$\frac{dm}{d\tau} + \lambda mA + \lambda R = 0 \quad . \tag{22}$$

Recalling that $A = d\theta/d\tau$, we can immediately write down the solution with zero initial rapidity in terms of θ:

$$m(\tau) = e^{-\lambda\theta(\tau)}m(\tau_i) \ - \ \lambda e^{-\theta(\tau)}\int_{\tau_i}^{\tau} e^{\lambda\theta(s)}R(s)\,ds \quad . \tag{23}$$

For $R \equiv 0$, corresponding to an uncharged rocket, we see that $m(\tau)$ decreases exponentially with $\theta(\tau)$. It also decreases exponentially with τ during the first period of uniformly accelerated motion, since in that period, $\theta(\tau) = (\tau - \tau_i - \delta)g + \theta(\tau_i + \delta)$. In particular, m can never vanish for an uncharged rocket. An uncharged rocket can uniformly accelerate forever, assuming that all of its rest mass can be used as fuel.

Now consider a charged rocket with R given by the Lorentz-Dirac expression $R(\tau) := -dA/d\tau$. Then (23) becomes:

$$m(\tau) = e^{-\lambda\theta(\tau)}m(\tau_i) \ - \ \lambda e^{-\theta(\tau)}\int_{\tau_i}^{\tau} e^{\lambda\theta(s)}\left(-\frac{dA}{ds}\right)\,ds \quad . \tag{24}$$

To dispel the notion that the charged rocket can uniformly accelerate for an arbitrarily long period without using any more fuel than would an uncharged rocket, we want to show that if A decreases monotonically from a constant value g down to 0 over a final proper-time interval $[\tau_f - \delta, \tau_f]$ of fixed length δ, then $m(\tau_f)$ must become negative for large τ_f. That is, for such an A and for a fixed initial mass $m(\tau_i)$, we cannot find positive-mass solutions defined for arbitrarily large proper times τ_f. This can be seen from the following simple estimates, in which it is helpful to remember that both λ and $-dA/ds$ are positive.[7]

First observe that from the Mean Value Theorem, for $\tau_f - \delta \le s \le \tau_f$,

$$\begin{aligned} \frac{\theta(\tau_f) - \theta(s)}{\delta} &\le \frac{\theta(\tau_f) - \theta(s)}{\tau_f - s} \\ &= \frac{d\theta}{d\tau}(\hat{\tau}) \quad \text{for some } \hat{\tau} \text{ with } s \le \hat{\tau} \le \tau_f \\ &= A(\hat{\tau}) \\ &\le g \quad . \end{aligned}$$

Using this, we have:

$$\begin{aligned} e^{-\lambda\theta(\tau_f)} \int_{\tau_f - \delta}^{\tau_f} e^{\lambda\theta(s)} \left(-\frac{dA}{ds}\right) ds &= \int_{\tau_f - \delta}^{\tau_f} e^{-\lambda(\theta(\tau_f) - \theta(s))} \left(-\frac{dA}{ds}\right) ds \\ &\ge e^{-\lambda g \delta} \int_{\tau_f - \delta}^{\tau_f} \left(-\frac{dA}{ds}\right) ds \quad , \\ &= e^{-\lambda g \delta}(-A(\tau_f) + A(\tau_f - \delta)) \quad , \\ &= e^{-\lambda g \delta} g \quad . \end{aligned} \tag{25}$$

Substituting (25) in (24), we see that to obtain a positive mass solution for arbitrarily large τ_f (and arbitrarily large radiated energy), we need arbitrarily great rocket mass (i.e., fuel) $m(\tau_i)$ to start with.

In other words, a charged rocket in Minkowski space which starts with a finite amount of fuel cannot uniformly accelerate for an arbitrarily long time, after which the acceleration is removed. Unlike a corresponding uncharged rocket, it must eventually run out of fuel. What is peculiar is that if it has sufficient fuel to get into the uniformly accelerated state, it will not run out of fuel until after the uniform acceleration is removed! It can uniformly accelerate for an arbitrarily long period, radiating all the while, but the physical contradiction of running out of fuel followed by the mass going negative will not be revealed until after the uniform acceleration is removed.

[7] That $-dA/ds$ is positive follows from the previous assumption, made for simplicity, that A decreases monotonically from g to 0. If we agree to eject mass at a positive rate (i.e., $dm/d\tau < 0$) until $A = 0$, and if we define τ_f to be the first time after deceleration that $A = 0$, then this assumption follows from (22) with $R := -dA/d\tau$.

We emphasize that this is a rigorous mathematical conclusion from the given assumptions—there are no approximations in the analysis which led to it. Physically, it is very hard to believe. The most questionable assumption seems to be the Lorentz-Dirac expression (17) for the radiated energy.

It is enlightening to follow the solution further to the final resting state at $\tau = 2\tau_f$, but before doing this let's think about what we would expect for an uncharged rocket. Since our formulation assumes that the exhaust velocity cannot be varied, the deceleration after $\tau = \tau_f$ is accomplished by taking in mass (and momentum), so we will have $dm/d\tau > 0$ for $\tau_f < \tau < 2\tau_f - \delta$. In effect, deceleration is accomplished by returning some of the previous exhaust energy-momentum to the rocket. For an uncharged rocket, the symmetry of the situation suggests that this energy-momentum return will be accomplished in time-symmetric fashion, and we can anticipate without calculation that all the exhaust energy-momentum will have been returned to the rocket at the final resting time $\tau = 2\tau_f$. In particular, the final rest mass should be the same as the initial rest mass. Indeed, this is what equation (24) does give if the radiation term $dA/d\tau$ is omitted.

However, the result is quite different for the charged rocket. In this case, $m(2\tau_f)$ differs from $m(\tau_i)$ by the amount of the second term containing the integral. We have $\theta(2\tau_f) = 0$, so the exponential factor in front of the integral doesn't contribute. The mass deficit at the end is

$$m(\tau_i) - m(2\tau_f) = \\ -\int_{\tau_i}^{\tau_i+\delta} e^{\lambda\theta(s)} \frac{dA}{d\tau}\, ds - \int_{\tau_f-\delta}^{\tau_f+\delta} e^{\lambda\theta(s)} \frac{dA}{d\tau}\, ds - \int_{2\tau_f-\delta}^{2\tau_f} e^{\lambda\theta(s)} \frac{dA}{d\tau}\, ds.$$

The first and third integrals are of moderate size, while the second integral over the interval $[\tau_f - \delta, \tau_f + \delta]$ is large for large τ_f because $e^{\lambda\theta}$ is large on this interval. In effect, the large initial-frame energy furnished over $[\tau_f - \delta, \tau_f + \delta]$ (corresponding to a small loss of rest mass at $\tau \approx \tau_f$ with high initial-frame velocity) has been transferred to the same large energy loss caused by a correspondingly large initial-frame rest mass loss.

To put it more physically, by observing his fuel gauge, the charged rocket pilot sees only a modest excess fuel loss over $[\tau_f - \delta, \tau_f + \delta]$ (relative to an uncharged rocket), but he *does* observe this loss, and he can figure out that because he is going very fast in the initial frame, it corresponds to a large initial-frame energy loss. Moreover, as he decelerates back to rest at $\tau = 2\tau_f$, this modest rest mass loss *grows* exponentially to an initial-frame excess rest mass loss large enough to pay for the radiated energy.

This last observation may sound strange, but properly viewed it is to be expected. The rest mass of an uncharged rocket, will increase exponentially during the period $[\tau_f + \delta, 2\tau_f - \delta]$ of uniform deceleration, and the same is true of the charged rocket. Over this period, the charged rocket behaves identically to an uncharged rocket *with the same rest mass at* $\tau = \tau_f + \delta$. However, the uncharged rocket which started with initial rest mass m_i at $\tau = 0$ does not have *exactly* the same rest mass at $\tau_f + \delta$ as the charged rocket with the same worldline and initial mass. There is a difference due to the radiation in the time interval $[0, \tau_f + \delta]$. This difference is modest even when the radiation is large. If τ_f is large enough to give large radiation, this modest difference in rest masses at $\tau = \tau_f + \delta$ is amplified by the exponential growth to a correspondingly large difference in rest masses at $\tau = 2\tau_f - \delta$.

This analysis provides additional insight into the discussion of Appendix 1. It demonstrates by explicit calculation that contrary to widely held beliefs, there is indeed *physical* radiation reaction for a particle which is uniformly accelerated for a finite time even though the Lorentz-Dirac radiation reaction expression vanishes identically during the period of uniform acceleration.[8] However, if we believe in the Lorentz-Dirac equation (and many experts don't), we must accept the very strange conclusion that all of this radiation reaction occurs at the beginning ($t \approx t_i$) and end ($t \approx t_f$) of the trip while the particle is being nudged into or out of its uniformly accelerated state.

Appendix 3: The field of a stationary particle in a static spacetime

It is often stated in the literature (e.g., [2]) that a charged particle which is stationary with respect to the coordinate frame in a static spacetime generates a pure electric field in that frame; since the Poynting vector vanishes, there is no radiation. However, we know of no proof in the literature, and the matter seems to us not as simple as it apparently does to the authors who make this assertion.

Implicit in such statements is that the field generated by the particle is the "retarded field" for its worldline. The problem is that there is no generally accepted, mathematically rigorous definition of "retarded field" in general spacetimes. In Minkowski space one can define the retarded field via the usual explicit formula, but no similar closed-form expressions are known for general spacetimes.

A retarded-field construction should be a rule which assigns to each charged particle worldline $\tau \mapsto z(\tau)$ (defined as a curve in spacetime with unit-norm tangent $u(\tau) := dz/d\tau$) a 2-form $F = F(x)$ satisfying Maxwell's equations with

[8]Whether there is radiation reaction for a perpetually uniformly accelerated particle depends on one's definition of "radiation reaction". The "radiation reaction" term in the Lorentz-Dirac equation does vanish identically, but there is no good physical reason to identify this term with physically observed radiation reaction. Instead, it seems more reasonable to obtain the answer for uniform acceleration for all time as a limit of whatever answer is eventually generally accepted for uniform acceleration for finite times. There is probably no reasonable way to do this without rejecting the Lorentz-Dirac equation, since the above answer for uniform acceleration for finite times (which is a consequence of the Lorentz-Dirac equation) is so strange.

source the distribution current associated with the worldline. Symbolically, these equations are

$$
\begin{aligned}
dF &= 0 \\
(*d*F)(x) &= \int \delta(x - z(\tau)) q u^\flat(\tau)\, d\tau \quad ,
\end{aligned}
$$

where d is the differential operator on alternating forms, $*$ the Hodge duality operation, δ the four-dimensional Dirac delta distribution, q the particle's charge, and $u^\flat$ the 1-form corresponding to u (see below).

To make the field "retarded", it is also required that the value of $F(x)$ at any spacetime point x off the worldline should depend only on the part of the worldline on or within the backward light cone with vertex x. In other words, any two worldlines which are identical inside this cone should yield the same $F(x)$.

Other assumptions might also reasonably be imposed. For example, one expects that for x off the worldline, the components of $F(x)$ would be an ordinary infinitely differentiable 2-form (*a priori* it is only a distribution). This assumption is not necessary for our purposes, but it does no harm and simplifies thought. One very plausible assumption which we shall need is that in a static spacetime, the retarded field for a stationary particle is time-independent.

Unfortunately, no mathematically rigorous retarded-field construction seems to be known for general spacetimes or even for static spacetimes. The discussion of Section 5.6 of [20], p. 220 gives the flavor of the mathematical difficulties.

Despite the lack of rigorous mathematical proof, most physicists seem prepared to believe that in any given spacetime, a unique retarded-field construction with the above properties ought to exist. Under this meta-mathematical assumption, we can show that the retarded field of a stationary particle in a static spacetime (6) is a pure electric field and that consequently the particle does not radiate. More precisely, there is no radiation through a stationary closed surface (stationary with respect to the "static" coordinates of (6)) surrounding the particle.

The idea is very simple. Given a retarded field, we can project out the electric part of it (relative to the static coordinates), and this projected electric part will still be "retarded". It is not obvious that it will satisfy Maxwell's equations (with the particle's worldline as source as above), but we shall show that it does. It follows that the electric part is also a retarded field.

If we believe in the uniqueness of the retarded field construction for the given spacetime, then this shows that the original retarded field was already a pure electric field. If we are not willing to make the uniqueness assumption, then at least we have shown that *there exists* a retarded field construction for static spacetimes for which the retarded field is pure electric and the particle does not radiate. If the retarded field construction is not unique, then we need additional physics to select the physically relevant retarded field in order to answer the question of whether a stationary charge radiates.

Now we prove the above assertion that the pure electric part of the retarded field for a stationary particle in a static spacetime is itself a solution of the above Maxwell's equations. As mentioned above, we assume that the retarded field is time-independent, and this is the only use of the "retarded field " assumptions. Thus we are really proving that the pure electric part of a time-independent solution is itself a solution.

Consider a particle stationary at the origin in a spacetime with the static metric (6). The four-velocity of the particle will be denoted $u(=u^i)$, and the corresponding index-lowered one-form as $u^\flat(=u_i := g_{i\alpha}u^\alpha)$. Explicitly, $u = g_{00}^{-1/2}\partial_{x_0}$, and $u^\flat = g_{00}^{1/2}dx^0$. Suppose we have a time-independent distribution 2-form $F = F_{ij}$ satisfying the Maxwell equations

$$\begin{aligned} dF &= 0 \\ *d*F &= -\delta_3 u^\flat \quad , \end{aligned} \tag{26}$$

where $*$ denotes the Hodge duality operation, d the differential operator on alternating forms, and $\delta_3(x,y,z) := \delta(x)\delta(y)\delta(z)$ is the three-dimensional Dirac delta distribution.[9] Time-independence means that the coefficients $F_{ij} = F_{ij}(x^1,x^2,x^3)$ do not depend on the coordinate time x^0.

We may uniquely write

$$F = \mathbf{E}^\flat \wedge u^\flat + \beta \quad , \tag{27}$$

where $\mathbf{E} = \sum_{I=1}^3 E^I \partial_{x_I}$ is a purely spatial vector field, $\mathbf{E}^\flat_i := g_{i\alpha}\mathbf{E}^\alpha$ the corresponding index-lowered 1-form, and β is a purely spatial 2-form. (We use bold-face for vectors in 4-space which are purely spatial with respect to the coordinate system used in (6), and we generally use capital Roman letters for space indices. All index lowering and raising is with respect to the spacetime metric rather than the Euclidean 3-space metric.) To say that β is purely spatial means that

$$\beta = \sum_{I,J=1}^3 \beta_{IJ} dx^I dx^J \quad .$$

Physically, β is the 3-space Hodge dual of the 1-form corresponding to the magnetic field vector $\mathbf{B}$. The proof of (27) follows routinely from expanding F as a linear combination of $dx^\alpha \wedge dx^\beta$, noting that u is proportional to dx^0, and collecting terms involving u.

We shall now show that if F satisfies (26), then the electric part $\mathbf{E}^\flat \wedge u^\flat$ of F also satisfies (26).

(a) Consider the first Maxwell equation $0 = dF = d(\mathbf{E}^\flat \wedge u) + d\beta$. We want to show that $d(\mathbf{E}^\flat \wedge u) = 0$. By routine calculation (directly, or cf. [5] ,

[9] Definitions of the differential-geometric quantities such as the Hodge dual can be found in [5], Chapter 2. The proof can be given within the rigorous framework of distribution theory, but we write it in the traditional physics language of Dirac delta "functions".

Section 5.4),

$$du^\flat = u^\flat \wedge a^\flat,$$

where $a := du/d\tau$ is the acceleration of a stationary observer. Hence

$$\begin{aligned} d(\mathbf{E}^\flat \wedge u^\flat) &= -d(u^\flat \wedge \mathbf{E}^\flat) = -du^\flat \wedge \mathbf{E}^\flat + u^\flat \wedge d\mathbf{E}^\flat \\ &= u^\flat \wedge (\mathbf{E}^\flat \wedge a^\flat + d\mathbf{E}^\flat) \quad . \end{aligned} \tag{28}$$

The point is that $d(\mathbf{E}^\flat \wedge u^\flat) = u^\flat \wedge (\textit{something})$ and hence is *not* a purely spatial 3-form. On the other hand, $d\beta$ *is* a purely spatial 3-form because its coefficients are time-independent by assumption. Hence $d(\mathbf{E}^\flat \wedge u^\flat)$ and $d\beta$ must separately vanish.

(b) Now we consider the other Maxwell equation $*d*F = \delta_3 u^\flat$ and try to prove that this can happen only if $*d*\beta = 0$. The 2-form β is purely spatial, so its Hodge dual $*\beta = u^\flat \wedge \mathbf{S}^\flat$ for some purely spatial vector $\mathbf{S}$. Apply the argument of part (a) with $\mathbf{S}$ in place of $\mathbf{E}$ to conclude that $d*\beta = u^\flat \wedge (\textit{something})$. Now take a Hodge dual to see that $*d*\beta$ is a purely spatial 1-form; i.e. $*d*\beta$ is the 1-form corresponding (under index-raising) to a vector orthogonal to u.

On the other hand, $*(\mathbf{E}^\flat \wedge u^\flat)$ is purely spatial with time-independent coefficients, hence $d*(\mathbf{E}^\flat \wedge u^\flat)$ is a purely spatial 3-form, hence $*d*(\mathbf{E}^\flat \wedge u^\flat)$ is a multiple of $u^\flat$. Thus we have

$$\delta_3 u^\flat = *d*(\mathbf{E}^\flat \wedge u^\flat) + *d*\beta$$

with the first term on the right a multiple of $u^\flat$ and the second term orthogonal to $u^\flat$; this can happen only if $*d*(\mathbf{E}^\flat \wedge u^\flat) = \delta^3 u^\flat$ and $*d*\beta = 0$.

This completes the proof that $\mathbf{E}^\flat \wedge u^\flat$ satisfies the Maxwell equations (26).

However, the field $\mathbf{E}$ is *not* usually a Coulomb field, contrary to impressions given by [2] and other authors.[10] To see that $\mathbf{E}$ is not necessarily a Coulomb field, consider a metric of the special form (8), for which (9) gives the acceleration as $a = (c'/c)\partial_x \neq 0$. A Coulomb field

$$\mathbf{C} := (x\partial_x + y\partial_y + z\partial_z)/(x^2 + y^2 + z^2)^{3/2}$$

would satisfy $d\mathbf{C}^\flat = 0$ except at the spatial origin (i.e. $\nabla \times \mathbf{C} = 0$ in 3-space), but this is inconsistent with the vanishing of (28) because

$$u^\flat \wedge \mathbf{C}^\flat \wedge a^\flat = (c'/c^2) dt \wedge dx \wedge \mathbf{C}^\flat \neq 0 \quad . \tag{29}$$

[10]This is probably more a question of language than of substance. For instance, although [2] states on page 172 that for the metric (10), "the accelerated observer ... only detects a Coulomb field", the expressions derived for the field are not precisely Coulomb fields in either the accelerated or Minkowski frames. Probably what was meant was something like "Coulomb-type" field.

Finally, we note that with $F := \mathbf{E}^\flat \wedge u^\flat$, the energy-momentum tensor (1) has $T^{0J} = 0$ for $J = 1, 2, 3$, which says that the Poynting vector vanishes and there is no radiation through any stationary closed surface surrounding the particle. This was worked out in [2] for the metric (8), and [8] obtains a special case of the same result in different language.

Acknowledgement: Because [2] is so widely cited (always favorably, to my knowledge), I chose to focus on it as representative of a point of view with which I have come to disagree. The scientific criticisms presented are in no way intended to denigrate its important contributions to our understanding of these issues. One such contribution is recognizing the role of the metric (10) as an interface between the physics of Minkowski space and the physics of more general spacetimes within which they can conveniently be compared. The present paper is deeply indebted to this advance.

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